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Benchmark Example No. 24

Free Vibration of a Under-critically Damped SDOF System

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VERiFiCATiON
BE24 Free Vibration of a Under-critically Damped SDOF System

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Front Cover

Volkstheater, Munich   Photo: Florian Schreiber

Overview	
Element Type(s):	SPRI, DAMP
Analysis Type(s):	DYN
Procedure(s):	TSTP
Topic(s):	
Module(s):	DYNA
Input file(s):	damped_sdof.dat

1 Problem Description

This problem consists of an under-critically damped linearly elastic SDOF system undergoing free vibrations, as shown in Fig. 1. The response of the system is determined and compared to the exact reference solution.

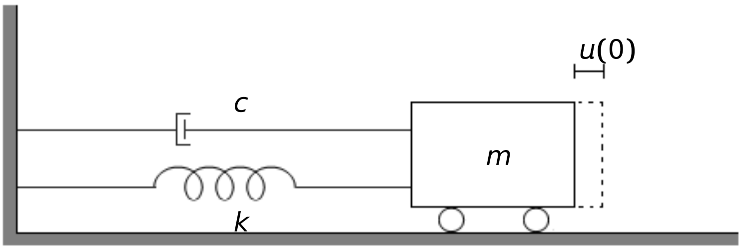

Figure 1: Problem Description

2 Reference Solution

The differential equation governing the free vibration of a linear elastic damped SDOF system, as shown in Fig. 1 is given by [1] [2]:

$$m\,\ddot{u} + c\,\dot{u} + k\,u = 0 \tag{1}$$

where c is the linear viscous damping, k the linear spring stiffness and m the mass of the system. Dividing Eq. 1 by m gives

$$\ddot{u} + 2\,\xi\,\omega_n\,\dot{u} + \omega_n^2\,u = 0 \tag{2}$$

where $\omega_n = \sqrt{k/m}$ as defined in Benchmark 23 and ξ represents the damping ratio

$$\xi = \frac{c}{2\,m\,\omega_n} = \frac{c}{c_{cr}} \tag{3}$$

The parameter c_{cr} is called the critical damping coefficient (Eq. 4), because it is the smallest value of c that inhibits oscillation completely. If c ¡ c_{cr} or ξ ¡ 1 the system is said to be under-critically damped and thus oscillates about its equilibrium position with a progressively decreasing amplitude [2].

$$c_{cr} = 2\, m\, \omega_n = 2\sqrt{k\, m} = \frac{2k}{\omega_n} \tag{4}$$

Free vibration is initiated by disturbing the system from its static equilibrium position by imparting the mass some displacement $u(0)$ and/or velocity $u(\dot{0})$ at time 0. Subjected to these initial conditions, the solution to the homogeneous differential equation of motion is:

$$u(t) = e^{-\xi\, \omega_n\, t} \left[u(0)\, cos\,(\omega_D\, t) + \left(\frac{\dot{u}(0) + \xi\, \omega_n\, u(0)}{\omega_D} \right) sin\,(\omega_D\, t) \right] \tag{5}$$

where ω_n represents the natural frequency of damped vibration and T_D the natural period of damped vibration given by

$$\omega_n = \omega_n\, \sqrt{1 - \xi^2} \tag{6}$$

$$T_d = \frac{2\pi}{\omega_D} = \frac{T_n}{\sqrt{1-\xi^2}} \tag{7}$$

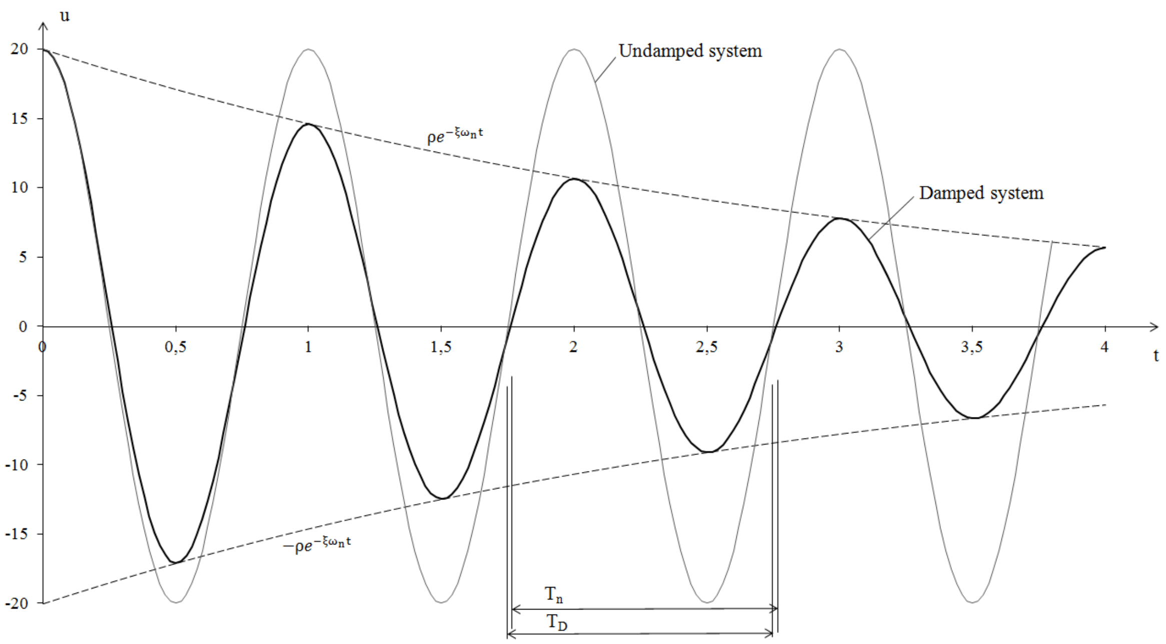

Figure 2: Effects of Damping on Free Vibration

The damped system oscillates with a displacement amplitude decaying exponentially with every cycle of vibration, as shown in Fig. 2. The envelope curves $\pm\rho e^{-\xi\,\omega_n\, t}$ touch the displacement curve at points slightly to the right of its peak values, where

$$\rho = \sqrt{u(0)^2 + \left(\frac{\dot{u}(0) + \xi\,\omega_n\,u(0)}{\omega_D}\right)^2} \tag{8}$$

3 Model and Results

The properties of the model are defined in Table 1. The system is initially disturbed from its static equilibrium position by a displacement of $20\ mm$ and is then let to vibrate freely. Eq. 5 is plotted in Fig. 3 and is compared to the calculated time history of the displacement of the SDOF system for different time integration methods. The time step is taken equal to $0.02\ sec$ corresponding to a dt/T ratio of $1/50$. From the curves, it is obvious that the examined integration schemes are in a good agreement with the exact solution. The damping of the SDOF system is represented in two ways, either by the spring element with a damping value in axial direction or with the damping element. The results obtained are exactly the same for both case. This can be visualised in the result files for the case of the Newmark integration scheme.

Table 1: Model Properties

Model Properties	Excitation Properties
$m = 1\,t$	$u(0) = 20\,mm$
$k = 4\pi^2\ kN/m$	$u(\dot{0}) = 0$
$T = 1\,sec$	
$\xi = 5\ \%$	

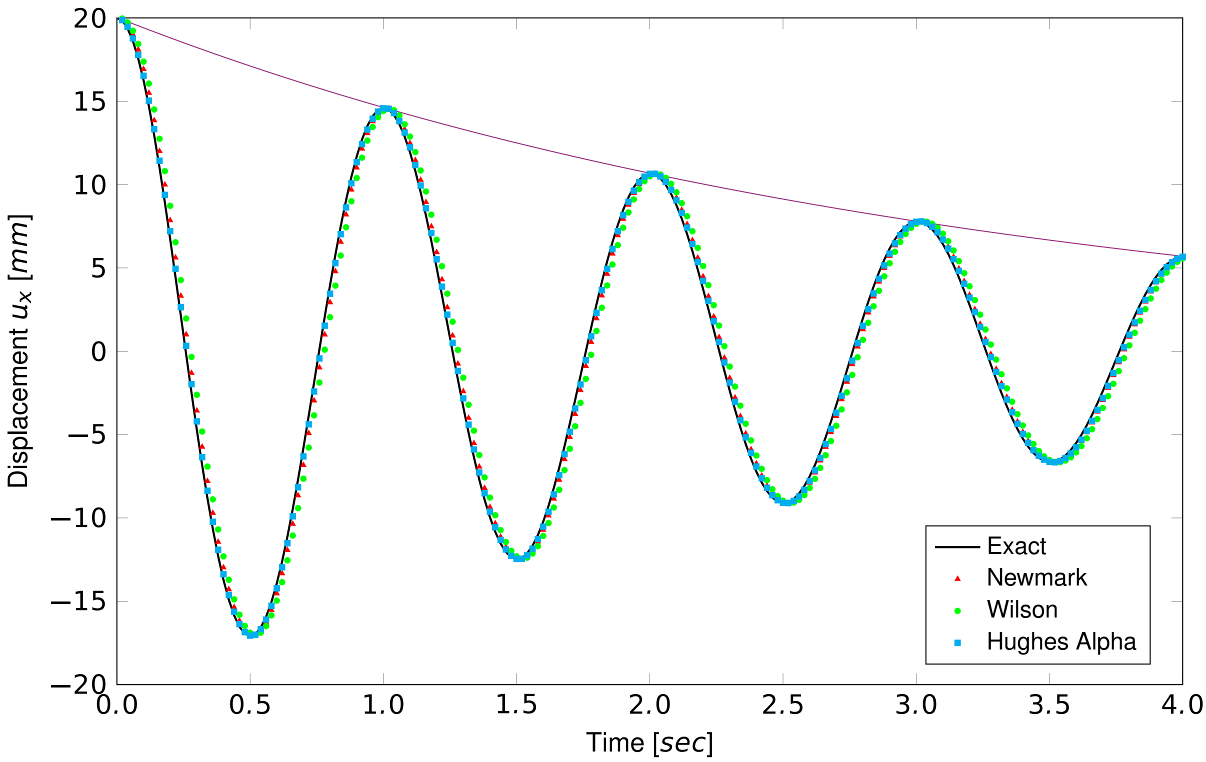

Figure 3: Damped Free Vibration Response

4 Conclusion

This example examines the response of a linear elastic under-critically damped SDOF system undergoing free vibration. It has been shown that the behaviour of the system is captured adequately.

5 Literature

[1] R. W. Clough and J. Penzien. *Dynamics of Structures*. 3rd. Computers & Structures, Inc., 2003.

[2] A. K. Chopra. *Dynamics of Structures: Theory and Applications to Earthquake Engineering*. Prentice Hall, 1995.